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Inclusions and inhomogeneities in electroelastic media with hexagonal symmetry

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Abstract. For a long time, the absence of explicit Green's functions (fundamental solutions) for electroelastic media has hindered progress in the modelling of the properties of piezoelectric materials. Michelitsch's recently derived explicit electroelastic Green's function for the infinite medium with hexagonal symmetry (transversely isotropic medium) [4] is used here to obtain *compact* closed-form expressions for the electroelastic analogue of the Eshelby tensor for spheroidal inclusions. This represents a key quantity for the material properties of piezoelectric solids and analysis of the related electroelastic fields in inclusions. For the limiting case of continuous fibers our results coincide with Levin's expressions [8]. The derived method is useful for an extension to non-spheroidal inclusions or inhomogeneities having an axis of rotational symmetry parallel to the hexagonal c-axis.

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1 Introduction

Within the last decades, the modelling of the properties of electroelastic coupled materials containing piezoelectric inclusions and inhomogeneities has become of great interest due to the increasing technological importance of such materials as sensors and actuators.

The elastic Eshelby tensor plays a fundamental role for purely elastic inclusion problems [1]. For electroelastic media a generalization exists [2,3], namely the electroelastic analogue of the Eshelby tensor (EAET). The EAET, which is a key quantity for the study of electroelastic fields in ellipsoidal inclusions, is an operator consisting of four tensors of fourth, third and second ranks.

Here our goal is to present *compact* explicit expressions for the components of the EAET for the case of spheroidal inclusions embedded in a medium with hexagonal symmetry having the same electroelastic characteristics. The treatment of the EAET of hexagonal material systems is desirable since many polycrystalline materials, such as uniaxial poled piezoelectric ceramics, show macroscopically hexagonal (transversely isotropic) symmetry where the poling axis then coincides with the hexagonal c-axis (axis of symmetry). In spite of the fact that many other piezoelectric materials have tetragonal or rhombohedral symmetry, one can describe their piezoelectric properties by assuming hexagonal symmetry (for only which the EAET can be obtained in closed form) as a very good approximation. For a medium with hexagonal symmetry, closed-form calculation of the EAET is feasible since the electroelastic Green's function that is required for this task is available for such a medium in explicit form [4].

To define this Green's function, we consider a homogeneous electroelastic material under isothermal conditions. The linear relations which govern such a material have the form

$$\begin{aligned}\sigma_{ij} &= \mathcal{C}_{ijkl}\epsilon_{kl} - e_{kij}E_k \\ D_i &= e_{ikl}\epsilon_{kl} + \eta_{ik}E_k.\end{aligned} \tag{1}$$

Here $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain tensors, $\boldsymbol{E}$ and $\boldsymbol{D}$ are the electric field and dielectric displacement vectors, respectively. $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}^E$ is the elastic moduli tensor for a fixed $\boldsymbol{E}$ vector, $\boldsymbol{\eta} = \boldsymbol{\eta}^{\varepsilon}$ indicates the permeability tensor for a fixed strain, and $\boldsymbol{e}$ denotes the tensor of piezoelectric constants which characterizes the related electroelastic effects. Relations (1) can conveniently be written in the following short form

$$\boldsymbol{J} = \mathbf{L}\boldsymbol{F},\ \boldsymbol{J} = \begin{pmatrix}\boldsymbol{\sigma}\\ \boldsymbol{D}\end{pmatrix}, \mathbf{L} = \begin{pmatrix}\boldsymbol{C} & \boldsymbol{e}\\ \boldsymbol{e}^T & -\boldsymbol{\eta}\end{pmatrix}, \boldsymbol{F} = \begin{pmatrix}\boldsymbol{\varepsilon}\\ -\boldsymbol{E}\end{pmatrix}. \tag{2}$$

The "matrix" $\mathbf{L}$ must be regarded as a linear operator which connects the tensor-vector pair $[\boldsymbol{\sigma}, \boldsymbol{D}]$ with the analogous pair $[\boldsymbol{\varepsilon}, \boldsymbol{E}]$. The superscript T denotes the transposition operation.
The field equations for the stress tensor $\boldsymbol{\sigma}$ (equation of equilibrium) and dielectric displacement vector $\boldsymbol{D}$ (equation of conservation of free electric charges) are given by

$$\partial_j\sigma_{ij} = -K_i, \quad \partial_l D_l = \rho_{\mathrm{e}} \tag{3}$$

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where $\boldsymbol{K}$ is the density of body forces and ρ_{e} is the density of free electric charges. Introducing the electric potential Φ and the elastic displacement field $\boldsymbol{u}$, the ansatz for the strain $\boldsymbol{\epsilon}$ and the electric field $\boldsymbol{E}$ is given by:

$$\begin{aligned} \epsilon_{ij} &= \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \\ E_i &= \qquad -\partial_i \Phi. \end{aligned} \tag{4}$$

Putting ansatz (4) using (1) into the field equations (3) we obtain a 4×4 differential equation of degree two for the field $\boldsymbol{\mathcal{U}} = (\boldsymbol{u}, \Phi)$ of the form:

$$\boldsymbol{\mathcal{T}}(\nabla)\boldsymbol{\mathcal{U}} + \boldsymbol{\mathcal{F}} = 0. \tag{5}$$

Here we introduced the gradient operator $\nabla = (\partial_i)$ ($i = 1,2,3$) with respect to the spatial coordinates $\boldsymbol{r} = (x,y,z)$. In (5) we have introduced the generalized force density $\boldsymbol{\mathcal{F}} = (\boldsymbol{K}, -\rho_{\mathrm{e}})$.

The symmetric 4×4 second-order differential operator $\boldsymbol{\mathcal{T}}(\nabla)$ can be written in the form

$$\boldsymbol{\mathcal{T}}(\nabla) = \begin{bmatrix} \boldsymbol{T}(\nabla) & \boldsymbol{t}(\nabla) \\ \boldsymbol{t}^T(\nabla) & \boldsymbol{\tau}(\nabla) \end{bmatrix} \tag{6}$$

where $\boldsymbol{T}(\nabla)$ is a 3×3 tensor operator that represents the elastic part. Its components are given by:

$$T_{ij}(\nabla) = \mathcal{C}_{ipjq}\partial_p\partial_q. \tag{7}$$

The vector operator (3×1 tensor) $\boldsymbol{t}(\nabla)$ has the components

$$t_i(\nabla) = e_{piq}\partial_p\partial_q \tag{8}$$

which represent the piezoelectric coupling. Finally the scalar operator

$$\tau(\nabla) = -\eta_{pq}\partial_p\partial_q \tag{9}$$

describes the dielectric part.

The paper is organized as follows: In Section 2 we give a brief outline of the theory of the electroelastic static Green's function of the infinite medium. There we introduce Michelitsch's explicit electroelastic Green's function of the infinite medium with hexagonal symmetry [4]. In Section 3 we derive using this Green's function the closed-form expressions for the components of the EAET for spheroidal inclusions or inhomogeneities. From these results we evaluate the limiting case of continuous fibers, which coincides with Levin's former result for the EAET [8].

2 Electroelastic Green's function

The vector field $\boldsymbol{\mathcal{U}}$ of equation (5) can be represented by the 4×4 electroelastic Green's function $\boldsymbol{\mathcal{G}}$ according to

$$\boldsymbol{\mathcal{U}}(\boldsymbol{r}) = \int \boldsymbol{\mathcal{G}}(\boldsymbol{r} - \boldsymbol{r}')\,\boldsymbol{\mathcal{F}}(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}'. \tag{10}$$

Consequently the 4×4 electroelastic Green's function has the general structure

$$\boldsymbol{\mathcal{G}}(\boldsymbol{r}) = \begin{pmatrix} \boldsymbol{G}(\boldsymbol{r}) & \boldsymbol{\gamma}(\boldsymbol{r}) \\ \boldsymbol{\gamma}^T(\boldsymbol{r}) & \boldsymbol{g}(\boldsymbol{r}) \end{pmatrix} \tag{11}$$

where $\boldsymbol{G}(\boldsymbol{r})$ is a second-rank tensor, $\boldsymbol{\gamma}(\boldsymbol{r})$ is a vector, and $\boldsymbol{g}(\boldsymbol{r})$ is a scalar function. It follows from (5) that the Green's function satisfies the equation

$$\boldsymbol{\mathcal{T}}(\nabla)\boldsymbol{\mathcal{G}}(\boldsymbol{r}) + \delta^3(\boldsymbol{r})\bar{\mathbf{I}} = 0, \quad \bar{\mathbf{I}} = \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{12}$$

with $\delta^3(\boldsymbol{r})$ representing the spatial δ-function, $\boldsymbol{I}$ the 3×3 and $\bar{\mathbf{I}}$ the 4×4 unit tensor.

The Green's function (11) has the following physical interpretation [4]:

$G_{mk}(\boldsymbol{r})$ is the elastic displacement in the x_m-direction at a space point $\boldsymbol{r}$ due to a unit point force at $\boldsymbol{r}' = 0$ in the x_k-direction ($m,k = 1,2,3$);
$\gamma_m(\boldsymbol{r})$ is the elastic displacement in the x_m-direction at a point $\boldsymbol{r}$ caused by a unit point charge at $\boldsymbol{r}' = 0$; ($m = 1,2,3$)
$\gamma_k^T(\boldsymbol{r})$ is the electric potential at point $\boldsymbol{r}$ caused by a unit point force at $\boldsymbol{r}' = 0$ in the x_k-direction; ($k = 1,2,3$)
$g(\boldsymbol{r})$ is the electric potential at point $\boldsymbol{r}$ caused by a unit point charge at $\boldsymbol{r}' = 0$.
The electroelastic Green's function (11) is symmetric, *i.e.* $\mathcal{G}_{pq} = \mathcal{G}_{qp}$ ($p,q = 1,2,3,4$).

Explicit evaluation of the Cartesian representation of the electroelastic Green's function (11) for the infinite medium with hexagonal symmetry, was obtained by Michelitsch in the form [4]

$$\boldsymbol{\mathcal{G}}(\boldsymbol{r}) = \sum_{l=1}^{4} \frac{1}{\sqrt{A_l\rho^2 + z^2}} \begin{pmatrix} g_{11}^{(l)} & g_{12}^{(l)} & g_{13}^{(l)} & g_{14}^{(l)} \\ g_{12}^{(l)} & g_{22}^{(l)} & g_{23}^{(l)} & g_{24}^{(l)} \\ g_{13}^{(l)} & g_{23}^{(l)} & g_{33}^{(l)} & g_{34}^{(l)} \\ g_{14}^{(l)} & g_{24}^{(l)} & g_{34}^{(l)} & g_{44}^{(l)} \end{pmatrix}. \tag{13}$$

Here we introduce the spatial Cartesian coordinates (x,y,z) with the z-axis being parallel to the hexagonal c-axis (axis of symmetry) and $\rho = \sqrt{x^2 + y^2}$. In (13) we have used the abbreviations:

$$g_{11}^{(l)} = \frac{1}{\mathcal{E}_l}\left[-\Gamma_{\mathrm{b}}(-A_l)\frac{x^2z^2 - y^2\left(A_l\rho^2 + z^2\right)}{\rho^4} + \Lambda_{\mathrm{b}\perp}(-A_l)\right] \tag{14}$$

$$g_{22}^{(l)} = \frac{1}{\mathcal{E}_l}\left[-\Gamma_{\mathrm{b}}(-A_l)\frac{y^2z^2 - x^2(A_l\rho^2 + z^2)}{\rho^4} + \Lambda_{\mathrm{b}\perp}(-A_l)\right] \tag{15}$$

$$g_{12}^{(l)} = \frac{1}{\mathcal{E}_l}\left[-\Gamma_{\mathrm{b}}(-A_l)\frac{xy(A_l\rho^2 + 2z^2)}{\rho^4}\right] \tag{16}$$

$$g_{13}^{(l)} = \frac{1}{\mathcal{E}_l}\left[-\Gamma_{\rm bc}\left(-A_l\right)\frac{xz}{\rho^2}\right] \tag{17}$$

$$g_{23}^{(l)} = \frac{1}{\mathcal{E}_l}\left[-\Gamma_{\rm bc}\left(-A_l\right)\frac{yz}{\rho^2}\right] \tag{18}$$

$$g_{33}^{(l)} = \frac{1}{\mathcal{E}_l}\Lambda_{\rm c}\left(-A_l\right) \tag{19}$$

$$g_{14}^{(l)} = \frac{1}{\mathcal{E}_l}\left[-\Gamma_{\rm b4}\left(-A_l\right)\frac{xz}{\rho^2}\right] \tag{20}$$

$$g_{24}^{(l)} = \frac{1}{\mathcal{E}_l}\left[-\Gamma_{\rm b4}\left(-A_l\right)\frac{yz}{\rho^2}\right] \tag{21}$$

$$g_{34}^{(l)} = \frac{1}{\mathcal{E}_l}\Lambda_{\rm c4}\left(-A_l\right) \tag{22}$$

$$g_{44}^{(l)} = \frac{1}{\mathcal{E}_l}\Lambda_{4}\left(-A_l\right) \tag{23}$$

and

$$\mathcal{E}_l = 4\pi\mathcal{C}_{66}A\prod_{j=1,(j\neq l)}^{4}\left(A_j - A_l\right) \tag{24}$$

with $A_1 = \mathcal{C}_{44}/\mathcal{C}_{66}$. The numbers A_2, A_3, A_4 are the zeros of the equation:

$$Aa^3 - Ba^2 + Ca - D = 0 \tag{25}$$

where the coefficients A, B, C, D are given by

$$A = -\eta_{11}\mathcal{C}_{11}\mathcal{C}_{44} - \mathcal{C}_{11}e_{15}^2 \tag{26}$$

$$\begin{aligned} B = &-\eta_{33}\mathcal{C}_{11}\mathcal{C}_{44} - \eta_{11}\left(\mathcal{C}_{11}\mathcal{C}_{33} - 2\mathcal{C}_{13}\mathcal{C}_{44} - \mathcal{C}_{13}^2\right) - \mathcal{C}_{44}e_{15}^2 \\ &- 2\mathcal{C}_{11}e_{15}e_{33} + 2\left(\mathcal{C}_{13} + \mathcal{C}_{44}\right)e_{15}\left(e_{31} + e_{15}\right) \\ &- \mathcal{C}_{44}\left(e_{31} + e_{15}\right)^2 \end{aligned} \tag{27}$$

$$\begin{aligned} C = &-\eta_{33}\left(\mathcal{C}_{11}\mathcal{C}_{33} - 2\mathcal{C}_{13}\mathcal{C}_{44} - \mathcal{C}_{13}^2\right) - \eta_{11}\mathcal{C}_{33}\mathcal{C}_{44} \\ &- 2e_{15}e_{33}\mathcal{C}_{44} - e_{33}^2\mathcal{C}_{11} + 2e_{33}\left(e_{31} + e_{15}\right) \\ &\times\left(\mathcal{C}_{13} + \mathcal{C}_{44}\right) - \mathcal{C}_{33}\left(e_{31} + e_{15}\right)^2 \end{aligned} \tag{28}$$

$$D = -\eta_{33}\mathcal{C}_{33}\mathcal{C}_{44} - e_{33}^2\mathcal{C}_{44}. \tag{29}$$

We furthermore introduced in (14)-(23) the quantities:

$$\Gamma_{\rm b}\left(a\right) = \frac{1}{a}\left[\Lambda_{\rm b\perp}\left(a\right) - \Lambda_{\rm b}\left(a\right)\right] \tag{30}$$

$$\Gamma_{\rm bc}\left(a\right) = \frac{1}{\sqrt{a}}\Lambda_{\rm bc}\left(a\right) \tag{31}$$

$$\Gamma_{\rm b4}\left(a\right) = \frac{1}{\sqrt{a}}\Lambda_{\rm b4}\left(a\right) \tag{32}$$

together with

$$\Lambda_{\rm b\perp}(a) = Aa^3 + Ba^2 + Ca + D \tag{33}$$

$$\begin{aligned} \Lambda_{\rm b}\left(a\right) = &-\left(\mathcal{C}_{66}a + \mathcal{C}_{44}\right)\Big[\left(\eta_{11}a + \eta_{33}\right)\left(\mathcal{C}_{44}a + \mathcal{C}_{33}\right) \\ &+ \left(e_{15}a + e_{33}\right)^2\Big], \end{aligned} \tag{34}$$

$$\begin{aligned} \Lambda_{\rm bc}\left(a\right) = &\sqrt{a}\left(\mathcal{C}_{66}a + \mathcal{C}_{44}\right)\Big[\left(e_{31} + e_{15}\right)\left(e_{15}a + e_{33}\right) \\ &+ \left(\eta_{11}a + \eta_{33}\right)\left(\mathcal{C}_{13} + \mathcal{C}_{44}\right)\Big], \end{aligned} \tag{35}$$

$$\begin{aligned} \Lambda_{\rm c}\left(a\right) = &-\left(\mathcal{C}_{66}a + \mathcal{C}_{44}\right)\Big[\left(\eta_{11}a + \eta_{33}\right)\left(\mathcal{C}_{11}a + \mathcal{C}_{44}\right) \\ &+ a\left(e_{31} + e_{15}\right)^2\Big], \end{aligned} \tag{36}$$

$$\begin{aligned} \Lambda_{\rm b4}\left(a\right) = &\sqrt{a}\left(\mathcal{C}_{66}a + \mathcal{C}_{44}\right)\Big[\left(\mathcal{C}_{13} + \mathcal{C}_{44}\right)\left(e_{15}a + e_{33}\right) \\ &- \left(\mathcal{C}_{44}a + \mathcal{C}_{33}\right)\left(e_{31} + e_{15}\right)\Big], \end{aligned} \tag{37}$$

$$\begin{aligned} \Lambda_{\rm c4}\left(a\right) = &-\left(\mathcal{C}_{66}a + \mathcal{C}_{44}\right)\Big[\left(\mathcal{C}_{11}a + \mathcal{C}_{44}\right)\left(e_{15}a + e_{33}\right) \\ &- a\left(\mathcal{C}_{13} + \mathcal{C}_{44}\right)\left(e_{31} + e_{15}\right)\Big], \end{aligned} \tag{38}$$

$$\begin{aligned} \Lambda_4(a) = &\left(\mathcal{C}_{66}a + \mathcal{C}_{44}\right)\Big[a^2\mathcal{C}_{11}\mathcal{C}_{44} + a\left(\mathcal{C}_{11}\mathcal{C}_{33} - 2\mathcal{C}_{13}\mathcal{C}_{44} - \mathcal{C}_{13}^2\right) \\ &+ \mathcal{C}_{33}\mathcal{C}_{44}\Big]. \end{aligned} \tag{39}$$

Here the 10 independent electroelastic moduli of a medium with hexagonal symmetry are introduced: five elastic moduli $\boldsymbol{C} = \{\mathcal{C}_{11}, \mathcal{C}_{13}, \mathcal{C}_{33}, \mathcal{C}_{44}, \mathcal{C}_{66}\}$, three piezoelectric constants $\boldsymbol{e} = \{e_{15}, e_{31}, e_{33}\}$ and two permeability coefficients $\boldsymbol{\eta} = \{\eta_{11}, \eta_{33}\}$. The explicit form (13) for the Green's function was derived using a method of integral transformation [5].

3 Inclusion problem

Inclusions of the same electroelastic characteristics that are considered here can be regarded as spatial domains which are allowed to undergo both uniform eigenstrain and -electric field. Physical examples where the assumption of uniform eigenstrains and -electric fields holds are domains containing phase transformation strains, induced eigenstrains by *e.g.* thermal expansion coefficient mismatch, and electric fields caused by a spontaneous polarization.

The EAET $\boldsymbol{\mathcal{S}}$ is then a linear operator which connects the induced fields $\boldsymbol{F}$ with a uniform eigenstrain and -electric field $\boldsymbol{F}^*$ *inside the (ellipsoidal) inclusion* according to [2,3,6]

$$\boldsymbol{F} = \boldsymbol{\mathcal{S}}\boldsymbol{F}^*. \tag{40}$$

A new proof is given in [6]. The crucial point that comes into play in the derivation of (40), and consequently to obtain the EAET in explicit form, is the evaluation of the $\boldsymbol{\mathcal{P}}$-operator, which is defined by the integral [2,3,6]

$$\int_V \nabla\boldsymbol{\mathcal{G}}(\boldsymbol{r} - \boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}' = \boldsymbol{\mathcal{P}}\cdot\boldsymbol{r}, \quad (\boldsymbol{r}\in V) \tag{41}$$

where V denotes the volume of the inclusion.

We consider in the following a spheroidal inclusion with semi-axes $(a_1 = a_2 = a, a_3)$ consisting of material with hexagonal symmetry which is embedded in a matrix of hexagonal symmetry with the same electroelastic characteristics. We assume that the semi-axis a_3 of the inclusion coincides with the c-axes (x_3-axis) of both the inclusion and the matrix. For an evaluation of integral (41) the explicit electroelastic Green's function $\boldsymbol{\mathcal{G}}(\boldsymbol{r})$ is needed from [4]. The $\boldsymbol{\mathcal{P}}$-operator generally takes the form

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} \boldsymbol{P} & \boldsymbol{H} \\ \boldsymbol{H}^T & \boldsymbol{p} \end{pmatrix}, \tag{42}$$

consisting of three tensors $\boldsymbol{P}, \boldsymbol{H}, \boldsymbol{p}$ of fourth-, third-, and second rank, reflecting the symmetries of the material characteristics, *i.e.* rank 4 of the elastic-, rank 3 of the piezoelectric-, and rank 2 of the dielectric tensor, respectively. The $\boldsymbol{\mathcal{P}}$-operator can be then conveniently written in the form

$$\boldsymbol{P} = P_1\mathbf{T}^2 + P_2\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + P_3(\mathbf{T}^3+\mathbf{T}^4) + P_5\mathbf{T}^5 + P_6\mathbf{T}^6 \tag{43}$$

$$\boldsymbol{H} = H_1\mathbf{U}^1 + H_2\mathbf{U}^2 + H_3\mathbf{U}^3 \tag{44}$$

$$\boldsymbol{p} = p_1\mathbf{t}^1 + p_2\mathbf{t}^2. \tag{45}$$

Here we introduced a useful tensor basis for hexagonal symmetry which is formed by the unit vector $\boldsymbol{m}$ parallel to the hexagonal c-axis (with components m_i) and by the projector $\theta_{ij} = \delta_{ij} - m_i m_j$ onto the plane perpendicular to the c-direction:

$$T^1_{ijkl} = \theta_{i(k}\theta_{l)j},\ T^2_{ijkl} = \theta_{ij}\theta_{kl},\ T^3_{ijkl} = \theta_{ij}m_k m_l,$$
$$T^4_{ijkl} = m_i m_j \theta_{kl}, \tag{46}$$

$$T^5_{ijkl} = \theta_{i)(k}m_{l)}m_{(j},\ T^6_{ijkl} = m_i m_j m_k m_l,\ U^1_{ijk} = \theta_{ij}m_k, \tag{47}$$

$$U^2_{ijk} = 2m_{(i}\theta_{j)k},\ U^3_{ijk} = m_i m_j m_k,\ t^1_{ij} = \theta_{ij},\ t^2_{ij} = m_i m_j. \tag{48}$$

The convenience of this tensorial basis is a consequence of the following properties: summing the products of the T-basis tensors over two indices forms a closed algebra. The sum of the products of the U-basis tensors over one index yields tensors of the T-basis and summing over two indices yields tensors of the t-basis. The t-basis is orthogonal with respect to contraction by one index, *i.e.* $t^r_{i\alpha}t^s_{\alpha j} = \delta_{rs}t^r_{ij}$ (no summing over r!).

The coefficients in (43), (44) and (45) are obtained as:

$$P_1 = -\frac{1}{8}\sum_{l=1}^{4}\frac{\Lambda_b(-A_l)}{\mathcal{E}_l}J_1^{(l)}, \tag{49}$$

$$P_2 = -\frac{1}{8}\sum_{l=1}^{4}\frac{\Lambda_b(-A_l)+\Lambda_{b\perp}(-A_l)}{\mathcal{E}_l}J_1^{(l)}, \tag{50}$$

$$P_3 = -\frac{1}{8}\sum_{l=1}^{4}\frac{\Gamma_{bc}(-A_l)}{\mathcal{E}_l}\left(J_1^{(l)} - \xi^2 A_l J_2^{(l)}\right), \tag{51}$$

$$P_5 = -\frac{1}{4}\sum_{l=1}^{4}\frac{1}{\mathcal{E}_l}\Big\{\left[\Lambda_c(-A_l)+\Gamma_{bc}(-A_l)\right]J_1^{(l)} + \left[\Lambda_b(-A_l)+\Lambda_{b\perp}(-A_l)-A_l\Gamma_{bc}(-A_l)\right]\xi^2 J_2^{(l)}\Big\}, \tag{52}$$

$$P_6 = -\frac{1}{2}\sum_{l=1}^{4}\frac{\Lambda_c(-A_l)}{\mathcal{E}_l}\xi^2 J_2^{(l)}, \tag{53}$$

$$H_1 = -\frac{1}{8}\sum_{l=1}^{4}\frac{\Gamma_{b4}(-A_l)}{\mathcal{E}_l}\left(J_1^{(l)} - \xi^2 A_l J_2^{(l)}\right), \tag{54}$$

$$H_2 = -\frac{1}{8}\sum_{l=1}^{4}\left[\frac{\Gamma_{b4}(-A_l)}{2\mathcal{E}_l}\left(J_1^{(l)} - \xi^2 A_l J_2^{(l)}\right) + \frac{\Lambda_{c4}(-A_l)}{\mathcal{E}_l}J_1^{(l)}\right], \tag{55}$$

$$H_3 = -\frac{1}{2}\sum_{l=1}^{4}\frac{\Lambda_{c4}(-A_l)}{\mathcal{E}_l}\xi^2 J_2^{(l)}, \tag{56}$$

$$p_1 = -\frac{1}{4}\sum_{l=1}^{4}\frac{\Lambda_4(-A_l)}{\mathcal{E}_l}J_1^{(l)}, \tag{57}$$

$$p_2 = -\frac{1}{2}\sum_{l=1}^{4}\frac{\Lambda_4(-A_l)}{\mathcal{E}_l}\xi^2 J_2^{(l)}. \tag{58}$$

Here $\xi = a/a_3$ denotes the ratio of the semi-axes of the inclusion. For *spheroidal* inclusions having an axis of rotational symmetry parallel to the hexagonal c-axis, the evaluation of (41) yields only two types of integrals coming into play in (49–58): These integrals are determined by the shape of the inclusion, thus we call them "*shape-integrals*". They have the form

$$J_1^{(l)} = 4\pi A_l\int_{-1}^{1}\frac{(1-u^2)\mathrm{d}u}{\left[1+(\xi^2-1)u^2\right]\left[A_l+(1-A_l)u^2\right]^{\frac{3}{2}}} = 8\pi\lambda_l^2\left[1 - \frac{1}{2}\xi^2 A_l\lambda_l\ln\left(\frac{\lambda_l+1}{\lambda_l-1}\right)\right],\quad l=1,2,3,4 \tag{59}$$

$$J_2^{(l)} = 4\pi \int_{-1}^{1} \frac{u^2 \mathrm{d}u}{[1+(\xi^2-1)u^2]\,[A_l + (1-A_l)u^2]^{\frac{3}{2}}}$$

$$= 8\pi\lambda_l^2 \left[\frac{1}{2}\lambda_l \ln\left(\frac{\lambda_l+1}{\lambda_l-1}\right) - 1\right], \quad l = 1,2,3,4 \quad (60)$$

where $\lambda_l = (1 - A_l\xi^2)^{-\frac{1}{2}}$. Equations (59) and (60) remain valid when λ_l is complex. The shape effects of the inclusion on the $\boldsymbol{\mathcal{P}}$-operator and consequently on the EAET are completely determined by these shape-integrals (59) and (60). These results can easily be extended to the case of a non-spheroidal inclusion or inhomogeneity having a shape of rotational symmetry with respect to the hexagonal c-direction. Then ξ in (59) and (60) becomes u-dependent, characterizing the shape of the inclusion.

The EAET $\boldsymbol{\mathcal{S}}$ defined in (40) is an operator with the general structure [6]

$$\boldsymbol{\mathcal{S}} = \begin{pmatrix} \mathbf{S} & -\mathbf{q} \\ \mathbf{Q} & -\mathbf{s} \end{pmatrix}. \quad (61)$$

Here we introduced the tensor $\mathbf{S}$ of rank 4, the tensors $\mathbf{Q}$ and $\mathbf{q}$ of rank 3, and the tensor $\mathbf{s}$ of rank 2. Using tensor basis (46–48) the EAET (61) assumes the form:

$$\mathbf{S} = S_1\mathbf{T}^2 + S_2\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + S_3\mathbf{T}^3 + S_4\mathbf{T}^4 + S_5\mathbf{T}^5 + S_6\mathbf{T}^6 \quad (62)$$

$$\mathbf{Q} = Q_1\mathbf{U}^{1T} + Q_2\mathbf{U}^{2T} + Q_3\mathbf{U}^{3T}, \;\; \mathbf{q} = q_1\mathbf{U}^1 + q_2\mathbf{U}^2 + q_3\mathbf{U}^3, \quad (63)$$

$$\mathbf{s} = s_1\mathbf{t}^1 + s_2\mathbf{t}^2. \quad (64)$$

Here we have introduced the abbreviations

$$S_1 = 2P_1(\mathcal{C}_{11} - \mathcal{C}_{66}) + P_3\mathcal{C}_{13} + H_1 e_{31}, \;\; S_2 = 2P_2\mathcal{C}_{66}, \quad (65)$$

$$S_3 = 2P_1\mathcal{C}_{13} + P_3\mathcal{C}_{33} + H_1 e_{33},$$
$$S_4 = 2P_3(\mathcal{C}_{11} - \mathcal{C}_{66}) + P_6\mathcal{C}_{13} + H_3 e_{31}, \quad (66)$$
$$S_5 = 2(P_5\mathcal{C}_{44} + 2H_2 e_{15}), \;\; S_6 = P_6\mathcal{C}_{33} + 2P_3\mathcal{C}_{13} + H_3 e_{33}, \quad (67)$$

$$Q_1 = 2H_1(\mathcal{C}_{11} - \mathcal{C}_{66}) + H_3\mathcal{C}_{13} + p_2 e_{31},$$
$$Q_2 = 2H_2\mathcal{C}_{44} + p_1 e_{15}, \quad (68)$$
$$Q_3 = 2H_1\mathcal{C}_{13} + H_3\mathcal{C}_{33} + p_2 e_{33}, \quad (69)$$

$$q_1 = H_1\eta_{33} - 2P_1 e_{31} - P_3 e_{33}, \;\; q_2 = H_2\eta_{11} - \frac{1}{2}P_5 e_{15}, \quad (70)$$

$$q_3 = H_3\eta_{33} - 2P_3 e_{31} - P_6 e_{33} \quad (71)$$
$$s_1 = p_1\eta_{11} - 2H_2 e_{15}, \;\; s_2 = p_2\eta_{33} - 2H_1 e_{31} - H_3 e_{33}. \quad (72)$$

We now consider the limiting case $a_3 \to \infty$ ($\xi \to 0$), corresponding to the *continuous cylindrical fiber*. In this limiting case the general expressions (49–58) assume the form

$$P_1 = -\pi \sum_{l=1}^{4} \frac{\Lambda_b(-A_l)}{\mathcal{E}_l},$$
$$P_2 = -\pi \sum_{l=1}^{4} \frac{\Lambda_{b4}(-A_l) + \Lambda_{b\perp}(-A_l)}{\mathcal{E}_l}, \quad (73)$$

$$P_3 = \pi \sum_{l=1}^{4} \frac{\Gamma_{bc}(-A_l)}{\mathcal{E}_l},$$
$$P_5 = -2\pi \sum_{l=1}^{4} \frac{1}{\mathcal{E}_l}\left[\Lambda_c(-A_l) + \Gamma_{bc}(-A_l)\right], \quad (74)$$

$$P_6 = 0, \quad H_1 = -\pi \sum_{l=1}^{4} \frac{\Gamma_{b4}(-A_l)}{\mathcal{E}_l},$$
$$H_2 = H_1 - 2\pi \sum_{l=1}^{4} \frac{\Lambda_{c4}(-A_l)}{\mathcal{E}_l}, \quad (75)$$

$$H_3 = 0, \quad p_1 = -2\pi \sum_{l=1}^{4} \frac{\Lambda_4(-A_l)}{\mathcal{E}_l}, \quad p_2 = 0. \quad (76)$$

These sums are evaluated in the appendix (Eqs. (A.11–A.13) and (A.19–A.23)). Taking into account these results we obtain

$$\mathbf{P} = -\frac{1}{4\mathcal{C}_{11}}\mathbf{T}^2 - \frac{1}{4}\left(\frac{1}{\mathcal{C}_{11}} + \frac{1}{\mathcal{C}_{66}}\right)\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) - \frac{1}{2}\left(\mathcal{C}_{44} + \frac{e_{15}^2}{\eta_{11}}\right)^{-1}\mathbf{T}^5 \quad (77)$$

$$\mathbf{H} = -\frac{e_{15}}{4(\eta_{11}\mathcal{C}_{44} + e_{15}^2)}\mathbf{U}^2, \quad \mathbf{p} = \frac{1}{2}\left(\eta_{11} + \frac{e_{15}^2}{\mathcal{C}_{44}}\right)^{-1}\mathbf{t}^1. \quad (78)$$

The components of the EAET for the continuous fiber are given by (62–64) together with the general relations (65–72) by using results (77–78). These results for the continuous fiber coincide with those obtained in a series of papers [7–9] employing other techniques.

4 Conclusions

Results (62–64) together with (65–72) represent the EAET in closed form and hold for spheroidal inclusions having rotational symmetry with respect to the hexagonal c-axis. The generalization of these results to non-spheroidal inclusions with such symmetry only modifies the shape-integrals (59) and (60).

Our results may be useful for a treatment of many inclusion- and inhomogeneity problems of the coupled electroelasticity and may stimulate future work.

Appendix

Here we consider expressions of the form

$$\alpha S_3+\beta S_2+\gamma S_1+\delta S_0=\sum_{l=1}^{4}\frac{p(-A_l)}{\mathcal{E}_l A_l}, \qquad (A.1)$$

where $p(-A_l)$ is a polynomial of *third*-order in $-A_l$

$$p(a)=\alpha a^3+\beta a^2+\gamma a+\delta. \qquad (A.2)$$

Let us now consider the function

$$h(a)=\frac{p(a)}{f(a)}=\frac{1}{4\pi A\mathcal{C}_{66}}\frac{p(a)}{(a+A_1)(a+A_2)(a+A_3)(a+A_4)}\cdot \qquad (A.3)$$

Since the numerator $p(a)$ is a third-order polynomial and the denominator $f(a)$ a fourth-order polynomial in a, we can decompose $h(a)$ in the form

$$h(a)=\sum_{l=1}^{4}\frac{h_l}{(a+A_l)}, \qquad (A.4)$$

where the constant coefficients h_l are given by

$$h_l=(a+A_l)h(a)|_{a=-A_l}. \qquad (A.5)$$

From this equation we obtain

$$h_l=\frac{p(-A_l)}{\mathcal{E}_l}, \qquad (A.6)$$

where we use the definition of the $\mathcal{E}_l$

$$\mathcal{E}_l=4\pi\mathcal{C}_{66}A\,\Pi^4_{j=1,j\neq l}(A_j-A_l). \qquad (A.7)$$

We thus can write

$$h(a)=\sum_{l=1}^{4}\frac{p(-A_l)}{\mathcal{E}_l(a+A_l)}=\frac{1}{4\pi A\mathcal{C}_{66}}\frac{p(a)}{(a+A_1)(a+A_2)(a+A_3)(a+A_4)}\cdot \qquad (A.8)$$

Equation (A.8) holds if $p(a)$ is a polynomial of at most *third* order in a. Putting $a=0$ in this equation we obtain

$$h(a=0)=\sum_{l=1}^{4}\frac{p(-A_l)}{\mathcal{E}_l A_l}=\frac{1}{4\pi A\mathcal{C}_{66}}\frac{\delta}{A_1A_2A_3A_4}=\frac{\delta}{4\pi\mathcal{C}_{44}D}, \qquad (A.9)$$

where $\delta=p(a=0)$ from equation (A.2) has been used. Thus only the term S_0 corresponding to the zero-order term in A_l contributes. The terms S_n corresponding to the powers A_l^n $(n=1,2,3)$ yield vanishing contributions. From equation (A.9) we therefore read off the property

$$\sum_{l=1}^{4}\frac{\alpha A_l^2+\beta A_l+\gamma}{\mathcal{E}_l}=0. \qquad (A.10)$$

From this equation, it follows that sums consisting of terms with quadratic functions of A_l in their numerators are vanishing. This holds for the sums

$$\sum_{l=1}^{4}\frac{\Gamma_b(-A_l)}{\mathcal{E}_l}=0, \qquad (A.11)$$

$$\sum_{l=1}^{4}\frac{\Gamma_{bc}(-A_l)}{\mathcal{E}_l}=0, \qquad (A.12)$$

$$\sum_{l=1}^{4}\frac{\Gamma_{b4}(-A_l)}{\mathcal{E}_l}=0. \qquad (A.13)$$

To evaluate the remaining sums with cubic functions of A_l in their numerators we have to evaluate the following sum:

$$\mathcal{H}=-\sum_{l=1}^{4}\frac{A_l^3}{\mathcal{E}_l}\cdot \qquad (A.14)$$

Because of the vanishing of (A.10) we can conclude that $\mathcal{H}$ can also be written in the more convenient form

$$\mathcal{H}=\sum_{l=1}^{4}\frac{(A_2-A_l)(A_3-A_l)(A_4-A_l)}{\mathcal{E}_l}\cdot \qquad (A.15)$$

This representation of $\mathcal{H}$ has the advantage that it can be evaluated in a straight-forward manner to arrive at

$$\mathcal{H}=\frac{(A_2-A_1)(A_3-A_1)(A_4-A_1)}{\mathcal{E}_1}=\frac{1}{4\pi\mathcal{C}_{66}A}, \qquad (A.16)$$

to obtain for (A.14)

$$\mathcal{H}=-\sum_{l=1}^{4}\frac{A_l^3}{\mathcal{E}_l}=\frac{1}{4\pi\mathcal{C}_{66}A}\cdot \qquad (A.17)$$

Here we have used the fact that in equation (A.15) the terms with $l=2,3,4$ are vanishing, thus only the term with $l=1$ contributes. Using (A.17) together with (A.10) we arrive at

$$-\sum_{l=1}^{4}\frac{\alpha A_l^3+\beta A_l^2+\gamma A_l+\delta}{\mathcal{E}_l}=\frac{\alpha}{4\pi\mathcal{C}_{66}A}, \qquad (A.18)$$

where $\alpha,\beta,\gamma,\delta$ denote arbitrary constants. Thus only the powers A_l^3 contribute to (A.18). Using (A.18) we read the remaining sums of equations (73–76) that contain terms with powers A_l^3 in their numerators to obtain

$$\sum_{l=1}^{4}\frac{\Lambda_b(-A_l)}{\mathcal{E}_l}=\frac{1}{4\pi\mathcal{C}_{11}}, \qquad (A.19)$$

$$\sum_{l=1}^{4}\frac{\Lambda_{b\perp}(-A_l)}{\mathcal{E}_l}=\frac{1}{4\pi\mathcal{C}_{66}}, \qquad (A.20)$$

$$\sum_{l=1}^{4} \frac{\Lambda_c(-A_l)}{\mathcal{E}_l} = \frac{1}{4\pi\left(\mathcal{C}_{44} + \dfrac{(e_{15})^2}{\eta_{11}}\right)}, \tag{A.21}$$

$$\sum_{l=1}^{4} \frac{\Lambda_{c4}(-A_l)}{\mathcal{E}_l} = \frac{e_{15}}{4\pi\left(\mathcal{C}_{44}\eta_{11} + (e_{15})^2\right)}, \tag{A.22}$$

$$\sum_{l=1}^{4} \frac{\Lambda_4(-A_l)}{\mathcal{E}_l} = \frac{-1}{4\pi\left(\eta_{11} + \dfrac{(e_{15})^2}{\mathcal{C}_{44}}\right)}\,. \tag{A.23}$$

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